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2002 J. Phys. A: Math. Gen. 35 5237

(http://iopscience.iop.org/0305-4470/35/25/306)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 5237–5253 PII: S0305-4470(02)31597-X

Neumann system, spherical pendulum and magnetic fields

Pavle Saksida

Department of Mathematics and Mechanics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Republic of Slovenia

E-mail: Pavle.Saksida@fmf.uni-lj.si

Received 4 December 2001, in final form 26 March 2002
Published 14 June 2002
Online at stacks.iop.org/JPhysA/35/5237

Abstract

In this paper we study a certain magnetic-like perturbation of the Neumann system. We prove the integrability of this system and show how its solutions are related to the solutions of a charged spherical pendulum influenced by the topologically nontrivial magnetic field $B_d(q) = q/\|q\|^3$ of the Dirac monopole. In the case when the quadratic potential of the Neumann system has a suitable axial symmetry, our system describes the motion of a charged particle under the influence of the potential and the homogeneous magnetic field $B_h(q) = (1, 0, 0)$.

PACS numbers: 45.20.Jj, 02.20.Sv
Mathematics Subject Classification: 37J15, 53D20, 70H06

1. Introduction

The Neumann system is one of the classical examples of integrable systems. In its original form, presented in [1], it describes a particle moving on the 2-sphere $S^2 = \left\{(q_1, q_2, q_3); q_1^2 + q_2^2 + q_2^3 = 1\right\}$ under the influence of the quadratic potential $V(q) = \langle q, A(q)\rangle$, where A is a symmetric 3×3 matrix with arbitrary eigenvalues. Suppose now that the particle is electrically charged and that its motion is additionally influenced by the magnetic-like field $B(q, \dot{q}) = f(q, \dot{q})(1, 0, 0)$, where $f(q, \dot{q})$ is equal to $\langle \dot{q} \times q, (1, 0, 0)\rangle$, that is to the $(1, 0, 0)$-component of the particle's angular momentum. The Lorentz-type force exerted on the particle by the field B is equal to $f(q, \dot{q}) \cdot (\dot{q} \times (1, 0, 0))$. This means that the Lorentz force of the homogeneous magnetic field $B_{hq} = (1, 0, 0)$ is amplified by the rotation of the particle around the $(1, 0, 0)$-axis. We shall study the Neumann system perturbed by the field B. In this paper the field B will be called the quasimagnetic field with the axis $(1, 0, 0)$. Physically more realistic systems, in which the motion of charged particles is influenced by the magnetic field affected by their own motion, are studied in the theory of magnetohydrodynamics. If in our situation the Neumann potential $V(q)$ is rotationally symmetric with respect to the

$(1, 0, 0)$-axis, our system describes the motion of the electrically charged particle under the influence of the potential $V(q)$ and the usual homogeneous magnetic field $B_h(q) = (1, 0, 0)$.

The first main result of the paper is theorem 2 proved in section 4. We will show that the Neumann system perturbed by the quasimagnetic field B is Arnold–Liouville integrable. Moreover, we shall give its Lax equation. To the author's knowledge, this adds a new example to the list of known integrable systems. Our system is closely related to the spherical pendulum which is another classical integrable system on S^2. It describes the motion of a particle on S^2 under the influence of the linear potential $W(q) = \langle q, l\rangle$, where $l \in \mathbb{R}^3$. In order to establish successfully the relationship between our quasimagnetic Neumann system and the spherical pendulum, we will modify the spherical pendulum by the magnetic field $B_d(q) = q/\|q\|^3$. This is the field of the Dirac magnetic monopole. Our construction shows how the field B_d of the monopole and the field B described above are related. In the simple, but nevertheless important rotationally symmetric case, this construction explains the relation between the magnetic monopole on S^2 and the physically more realistic homogeneous magnetic field $B_h(q)$ on the same space.

Systems with magnetic fields can be described in terms of the Kaluza–Klein construction. This amounts to adding to the configuration space a cyclic coordinate whose conjugate momentum is the conserved charge. In symplectic terms, the Kaluza–Klein procedure is an example of the symplectic reconstruction—a process which is inverse to the symplectic reduction. Symplectic reconstruction and geometric phases were studied in [2–4]. We will show that in the case of the spherical pendulum with the magnetic monopole, the Kaluza–Klein construction yields the system which describes the motion of a particle on the 3-sphere $S^3 = \{g = (q_1, q_2, q_3, q_4)\}$ under the influence of the potential $U(q) = \langle q, \tilde{A}(q)\rangle$ such that the eigenvalues of $\tilde{A}$ are $\{a, a - a, -a\}$. This is a special case of the Neumann system on S^3. In other words, we will show that the Neumann system on S^3 with the potential $U(g)$ is the symplectic reconstruction of the magnetic spherical pendulum. A similar construction, relating a system with magnetic monopole on $\mathbb{CP}^n$ to a system with quartic potential on S^{2n+1}, was given in [5]. If we now project the above Neumann system on S^3 to the equatorial 2-sphere $S^2 = \{(q_1, 0, q_3, q_4)\}$ in S^3 in a suitable way, we obtain a quasimagnetic perturbation of the Neumann system on S^2. This system describes the motion of a charged particle under the influence of the potential $V(q) = \langle q, A(q)\rangle$ and the quasimagnetic field $B(q, \dot{q}) = f(q, \dot{q})(1, 0, 0)$.

The above construction can be made more precise. We will show that the Hamiltonian formulation of the Neumann system on S^2 with the potential $V(q)$ and the quasimagnetic field B is (T^*S^2, ω_c, H_m), where

$$H_m(q, p_q) = \tfrac{1}{2}\|p_q \times q - (P + \langle p_q \times q, \sigma\rangle)\sigma\|^2 + V(q) \tag{1}$$

$\sigma = (1, 0, 0)$ and P is a real constant. Suppose that in our coordinates the potential $V(q)$ has the expression $V(q) = (\lambda_1 + d)q_1^2 + (d - \lambda_1)\big(q_2^2 + q_3^2\big) - 2\lambda_3 q_1 q_2 + 2\lambda_2 q_1 q_3$, where d, λ_1, λ_2 and λ_3 are arbitrary. Let $l = (\lambda_1, -\lambda_3, \lambda_2)$. In theorem 2 we shall see that $F\colon T^*S^2 \to \mathbb{R}$, given by

$$F(q, p_q) = \langle p_q \times q, l\rangle + (\langle p_q \times q, \sigma\rangle + P)V(q) \tag{2}$$

is an integral of our system, which proves the Arnold–Liouville integrability of (T^*S^2, ω_c, H_m).

Let $(T^*S^2, \omega_c + P\omega_d, H_{sp})$ be the spherical pendulum with the charge P in the magnetic field of the Dirac monopole. The magnetic field B_d is encoded by the form ω_d which is the pull-back to T^*S^2 of the volume form on S^2. The Hamiltonian is $H_{sp}(q, p_q) = \frac{1}{2}\|p_q\|^2 + \langle q, l\rangle$. We will see that an integral of this system is $G(q, p_q) = \langle p_q \times q, l\rangle + P\langle q, l\rangle$. Let us now equip the 2-sphere $S^2 = \big\{q = (q_1, q_2, q_3);\ \sum_{i=1}^3 q_i^2 = 1\big\}$ with the spherical coordinates

$q = (q_1, q_2, q_3) \mapsto (q_1, q_2 + \mathrm{i}q_3) = (\cos\vartheta, \mathrm{e}^{\mathrm{i}\varphi}\sin\vartheta)$. Let $\flat\colon T_qS^2 \to T_q^*S^2$ be given by $\flat(X) = X^\flat = \langle X, -\rangle_q$. We will prove the following theorem which is our second main result.

Theorem 1. *Let $(T^*S^2, \omega_c + P\omega_d, H_{sp})$ be the magnetic spherical pendulum with the gravitational force directed along the arbitrarily chosen vector $l \in \mathbb{R}^3$. Let the curve $(Q(t), P_Q(t))\colon [c, d] \to T^*S^2$ be a solution of this system such that $G(Q(t), P_Q(t)) = C$ for every $t \in [c, d]$. If in spherical coordinates*

$$Q(t) = (\cos(\vartheta(t)), \mathrm{e}^{\mathrm{i}\varphi(t)}\sin(\vartheta(t)))\colon [c, d] \to S^2$$

then the curve

$$q(t) = \left(\cos\left(\tfrac{1}{2}\vartheta(t)\right), \mathrm{e}^{\mathrm{i}(\varphi(t)-\frac{\pi}{2})}\sin\left(\tfrac{1}{2}\vartheta(t)\right)\right)\colon [c, d] \to S^2$$

is the solution of the quasimagnetic Neumann system (T^*S^2, ω_c, H_m) such that

$$F(q(t), p_q(t)) = F(q(t), (\dot q(t))^\flat) = C \qquad t \in [c, d]$$

where H_m is given by (1) and F is the integral given by (2).

We note that, even when the curve $Q(t)$ is a circle with axis l, the curve $q(t)$ has no symmetry with respect to l whenever l is not parallel to $(1, 0, 0)$. Theorem 1 is an immediate corollary of propositions 3 and 7 proved below. The key ingredient of our construction is the relation between two different representations of S^2. When considered as the configuration space of the spherical pendulum, the sphere S^2 will be represented as an adjoint orbit in the Lie algebra $\mathfrak{su}(2)$. The configuration space of the Neumann system will, in turn, be the Cartan model of S^2 in the Lie group $SU(2)$, i.e. the fixed-point set of a suitable involutive anti-isomorphism of $SU(2)$. Both models are naturally related to $S^3 = SU(2)$. The adjoint orbit is the base space of the Hopf fibration $SU(2) \to S^2$ given by $g \mapsto \mathrm{Ad}_g(\sigma)$ for some $\sigma \in \mathfrak{su}(2)$, and the Cartan model is a totally geodesic submanifold in $SU(2)$. This constellation of spheres will yield the relation between the spherical pendulum and the perturbed Neumann system, and in particular between the magnetic fields B_d and B_h and the quasimagnetic field B. The magnetic field B_h is given by an exact 2-form and B is closely related to B_h. Typically such fields are less symmetric than those given by the topologically nontrivial forms. Examples of integrable systems with exact magnetic and gyrostatic terms have been studied by many authors. A classical example is the paper [6] of Volterra. Such forms arise in the study of the motion of a heavy solid in the fluid, see e.g. in [7]. An important example is also the Kowalevski top with the gyrostatic term described in [8, 9]. Recently, another integrable system of this kind was found by Sokolov in [10].

The paper is divided into five sections. In section 2 we describe the special case of the Neumann system on S^3 mentioned above. Section 3 is devoted to the spherical pendulum with the Dirac magnetic monopole and the connection of this system with the special Neumann system on S^3. This gives the first part of the proof of theorem 1. In section 4 we explain the relation between the special Neumann system on S^3 and the quasimagnetic perturbation of the Neumann system on S^2. The first part of the section which includes propositions 4 and 5 concentrates on the construction and description of the quasimagnetic perturbation of the Neumann system. In the rest of the section we prove the integrability of the new quasimagnetic system and conclude the proof of theorem 1. We also describe the relation between the magnetic spherical pendulum and the axially symmetric Neumann system on S^2 perturbed by the homogeneous magnetic field B_h. We summarize the paper and suggest possible directions for further research in section 5.

2. A special case of the Neumann system on S^3

The Neumann system on $S^3 = \left\{g = (q_1, q_2, q_3, q_4);\ \sum_{i=1}^4 q_i^2 = 1\right\}$ describes the motion of a particle on S^3 under the influence of the potential $U(g) = \langle g, A(g)\rangle$, where A is a symmetric matrix. In suitable coordinates we have $U(g') = \sum_{i=1}^4 a_i (q_i')^2$. Therefore, this system can also be viewed as a description of the motion of four one-dimensional harmonic oscillators with positions q_i' $(i = 1, \ldots, 4)$, subject to the constraint $\sum_{i=1}^4 (q_i')^2 = 1$. Thus, the constants a_i will be called the spring constants of the system. We shall be interested in the Neumann system on S^3 in which the spring constants satisfy the equations $a_1 = a_2$, $a_3 = a_4$ and $a_1 = -a_3$. Let us identify the sphere S^3 with the special unitary group $SU(2)$ via the map

$$g = (q_1, q_2, q_3, q_4) \mapsto g = \begin{pmatrix} q_1 + \mathrm{i}q_2 & q_3 + \mathrm{i}q_4 \\ -q_3 + \mathrm{i}q_4 & q_1 - \mathrm{i}q_2 \end{pmatrix}. \tag{3}$$

Let ω_c denote the canonical cotangent symplectic structure on the cotangent bundle $T^*SU(2)$ and let $\sigma = \mathrm{diag}(\mathrm{i}, -\mathrm{i}) \in \mathfrak{su}(2)$.

Proposition 1. *The Neumann system on the 3-sphere whose spring constants are* $\{a, a, -a, -a\}$ *can be expressed as the Hamiltonian system* $(T^*SU(2), \omega_c, H)$*, where*

$$H(g, p_g) = \tfrac{1}{2}\|p_g\|^2 + \langle \lambda, \mathrm{Ad}_g(\sigma)\rangle$$

for any $\lambda \in \mathfrak{su}(2)$ *such that* $\|\lambda\|^2 = -\frac{1}{2}\mathrm{Tr}(\lambda^2) = a^2$.

Proof. Let $\lambda = \mathrm{i}(\lambda_1\sigma_1 + \lambda_2\sigma_2 + \lambda_3\sigma_3)$, where σ_j are the standard Pauli spin matrices. For $g \in SU(2)$ we have $g^{-1} = g^*$. The matrix multiplication and identification (3) give

$$\langle \lambda, \mathrm{Ad}_g(\sigma)\rangle = \lambda_1\left(q_1^2 + q_2^2 - q_3^2 - q_4^2\right) + 2\lambda_2(q_1q_3 - q_2q_4) + 2\lambda_3(q_1q_4 + q_2q_3).$$

It is now straightforward to check that this quadratic form has two double eigenvalues $a_{1,2} = +\|\lambda\|$ and $a_{3,4} = -\|\lambda\|$, which proves the proposition. □

It is well known that the Neumann system is completely integrable, see e.g. [11, 12]. Since we shall need a specific form of the integrals, let us nevertheless outline the proof of the integrability of the system $(T^*SU(2), \omega_c, H)$.

Proposition 2. *The Hamiltonian system* $(T^*SU(2), \omega_c, H)$ *is completely integrable. A set of Poisson-commuting integrals is given by*

$$M(g, p_g) = \left\langle p_g g^{-1}, \mathrm{Ad}_g(\sigma)\right\rangle \qquad E(g, p_g) = \left\langle p_g g^{-1}, \lambda\right\rangle \qquad \text{and} \qquad H \tag{4}$$

where $p_g g^{-1}$ *denotes the adjoint of the right translation.*

Proof. The Legendre transformation and the fact that $\|p_g\| = \|g_t g^{-1}\|$ tell us that the Lagrangian of our system is

$$\mathcal{L}(g(t)) = \int_c^d \left(\frac{1}{2}\left\|g_t g^{-1}\right\|^2 - \langle \lambda, \mathrm{Ad}_g(\sigma)\rangle\right) \mathrm{d}t.$$

Let $g(t)\colon [c, d] \to SU(2)$ be a path such that $g(c) = g_1$ and $g(d) = g_2$, and let $a(t, s)\colon [c, d] \times (-\epsilon, \epsilon) \to SU(2)$ be a map such that $a(t, 0) \equiv g(t)$, $a(c, s) \equiv g_1$ and $a(d, s) \equiv g_2$. Let $g(t, s) = a(t, s)g(t)$. Then a calculation, in which one uses integration by parts, the relation $\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{Ad}_{g(u)}(\kappa) = [g_u g^{-1}, \mathrm{Ad}_g(\kappa)]$ and the ad-invariance of the Killing form, $\langle \xi, [\eta, \zeta]\rangle = \langle [\xi, \eta], \zeta\rangle$, give

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathcal{L}(g(t, s)) = \int_c^d \left\langle \left(g_t g^{-1}\right)_t - [\lambda, \mathrm{Ad}_g(\sigma)], \delta a\right\rangle \mathrm{d}t$$

where $\delta a = \frac{d}{ds}a(t,s)a^{-1}(t,s)\big|_{s=0}$. Thus, the equation of motion of our system is

$$\left(g_t g^{-1}\right)_t = [\lambda, \mathrm{Ad}_g(\sigma)]. \tag{5}$$

It is straightforward to check that equation (5) is equivalent to the Lax equation

$$L_t = [A, L]$$

where

$$L(z) = \mathrm{Ad}_g(\sigma) + z g_t g^{-1} + z^2\lambda \qquad A(z) = g_t g^{-1} + z\lambda.$$

Therefore, the conserved quantities of equation (5) are the coefficients of the characteristic polynomial $\det(L(z) - wI)$. Since $\mathrm{Tr}(L(z)) = 0$, the integrals are the coefficients of the polynomial $\det(L(z)) = \langle L(z), L(z)\rangle$ which is equal to

$$\|\sigma\|^2 + z\left\langle g_t g^{-1}, \mathrm{Ad}_g(\sigma)\right\rangle + z^2\left(\left\|g_t g^{-1}\right\|^2 + 2\langle\lambda, \mathrm{Ad}_g(\sigma)\rangle\right) + z^3\left\langle g_t g^{-1}, \lambda\right\rangle + z^4\|\lambda\|^2. \tag{6}$$

If we use the identification $p_g = \langle g_t, -\rangle_g$, where $\langle -, -\rangle_g$ is the bi-invariant metric on $SU(2)$ defined by the Killing form, then $g_t g^{-1}$ corresponds to $p_g g^{-1}$. Formula (6) tells us that, in addition to the Hamiltonian H, the functions $M, E \colon T^*SU(2) \to \mathbb{R}$ given by (4) are indeed integrals of our system. We shall include the proof of Poisson-commutativity of M and E in the proof of proposition 3. For a more general proof we refer the reader to the seminal work [12], and for different proofs to [13] and [5]. □

3. Spherical pendulum with the Dirac monopole

In this section we shall study the relation between the special Neumann system $(T^*SU(2), \omega_c, H)$ and the spherical pendulum with the additional magnetic field caused by the Dirac monopole. This relation will give us an interesting physical interpretation of the integrals $M, E \colon T^*SU(2) \to \mathbb{R}$ constructed above. The geometry of the spherical pendulum was studied by Duistermaat in [14]. The pendulum moving in the field of the magnetic monopole is described, e.g. in [15, 16]. An interesting connection of this system with the configurations of vortex filament is given in [17].

A result similar to the one described in this section is already implicit in the existing literature. Felix Klein showed in [18] that the magnetic spherical pendulum is a symplectic reduction of the Lagrange top. The phase space of this (and indeed of any) top is $T^*SO(3) = T^*\mathbb{RP}^3$. In [5] we describe how a Hamiltonian system on $T^*\mathbb{RP}^n$ can be pulled back to a Hamiltonian system on T^*S^n via the antipodal map. In particular we note that a polynomial potential of degree n on $\mathbb{RP}^n$ pulls back to a polynomial potential of degree $2n$ on S^n. If we apply this procedure on the Lagrange top, we obtain a system on $T^*SU(2) = T^*S^3$ similar to the Neumann system with a circular symmetry. The difference between these two systems is in the kinetic energy. In the case of the Neumann system it is given by the ad-invariant Killing form on $\mathfrak{su}(2)$, while on the lifted Lagrange top it is determined by a metric on $\mathfrak{su}(2)$ which is only left-invariant.

We recall that the spherical pendulum is the Hamiltonian system $(T^*S^2, \omega_c, H_{sp})$, where

$$H_{sp}(Q, P_Q) = \tfrac{1}{2}\|P_Q\|^2 + \langle Q, L\rangle \tag{7}$$

and $Q = (Q_1, Q_2, Q_3) \in S^2 \subset \mathbb{R}^3$. The vector $L = (L_1, L_2, L_3) \in \mathbb{R}^3$ is the direction of the gravitational force. A frequent choice in the literature is $L = (0, 0, 1)$ which sets the potential function $\langle L, Q\rangle$ to Q_3. Let $\tilde{\omega}_d$ be the volume 2-form on S^2 and let $\omega_d = \pi^*(\tilde{\omega}_d)$, where $\pi \colon T^*S^2 \to S^2$ is the natural projection. The Hamiltonian system $(T^*S^2, \omega_c + P\omega_d, H_{sp})$ describes the motion of a particle with the electric charge P confined to the 2-sphere under the influence of the gravitational potential $\langle Q, L\rangle$ and of the monopole magnetic field $B_d(Q) = Q/\|Q\|^3$. For a detailed explanation of the relation between the form ω_d and the magnetic field B_d, see [4].

From proposition 2 it is clear that the special Neumann system is invariant with respect to the suitable $U(1)$-action. More precisely, let ρ be the action of the subgroup $U_\sigma(1) = \{\exp(s\sigma); s \in [0, 2\pi)\}$ on $SU(2)$ given by $\rho_u(g) = g \cdot u$ and let $\tilde{\rho}$ be the lifting of ρ to the cotangent bundle $T^*SU(2)$. Then the system $(T^*SU(2), \omega_c, H)$ is invariant with respect to the action $\tilde{\rho}$. Let $\mathcal{S}^2 \subset \mathfrak{su}(2)$ be the adjoint orbit of σ. Consider the map

$$f : SU(2) \to S^2 = \mathcal{S}^2 \subset \mathfrak{su}(2) \qquad f(g) = Q = \mathrm{Ad}_g(\sigma).$$

In spherical coordinates this map has the expression

$$\begin{pmatrix} e^{i\psi}\cos\vartheta & e^{i\varphi}\sin\vartheta \\ -e^{-i\varphi}\sin\vartheta & e^{-i\psi}\cos\vartheta \end{pmatrix} \stackrel{f}{\longmapsto} \begin{pmatrix} i\cos 2\vartheta & e^{i(\varphi+\psi+\frac{\pi}{2})}\sin 2\vartheta \\ -e^{-i(\varphi+\psi+\frac{\pi}{2})}\sin 2\vartheta & -i\cos 2\vartheta \end{pmatrix} \tag{8}$$

The fibres of f are precisely the orbits of ρ and f is a realization of the well-known Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$. Let

$$T_g SU(2) = \mathrm{Vert}_g \oplus \mathrm{Hor}_g \cong \mathrm{span}(\mathrm{Ad}_g(\sigma)) \oplus \mathrm{span}([X, \mathrm{Ad}_g(\sigma)]; X \in \mathfrak{su}(2)) \tag{9}$$

be the decomposition of the tangent space, where Vert_g is the tangent space of the fibre of f and Hor_g its orthogonal complement with respect to the Riemannian metric on $SU(2) = S^3$ determined by the Killing form. The second sum is obtained from the first one by identifying $T_g SU(2)$ with $\mathfrak{su}(2)$ via the right translations by g^{-1}. Accordingly, we have the decomposition

$$T_g^* SU(2) = \mathrm{Vert}_g^* \oplus \mathrm{Hor}_g^* \qquad p_g = p_g^v + p_g^h \qquad p_g^v \in \mathrm{Vert}_g^* \quad p_g^h \in \mathrm{Hor}_g^* \tag{10}$$

where Vert_g^* is the annihilator of Hor_g and Hor_g^* is the annihilator of Vert_g. Observe that the restriction $(Df)_g$: $\mathrm{Hor}_g \to T_{f(g)}\mathcal{S}^2$ is an isometry for every $g \in SU(2)$. We can lift the map f to the map $\mathcal{F}$: $T^*SU(2) \to T^*S^2$ by setting

$$\mathcal{F}(g, p_g) = \left(\mathrm{Ad}_g(\sigma), \left\{p_g g^{-1}, \mathrm{Ad}_g(\sigma)\right\}\right) = (Q, P_Q) \tag{11}$$

where $\langle\{p_g g^{-1}, X\}, Y\rangle = \langle p_g g^{-1}, [X, Y]\rangle$ for every $X, Y \in \mathfrak{su}(2)$.

Now we shall describe a decomposition of the canonical form ω_c on $T^*SU(2)$ induced by the map $\mathcal{F}$. First we have the decomposition of the tautological form α on $T^*SU(2)$:

$$\alpha_{(g,p_g)}\left(X_g, X_{p_g}\right) = p_g(X_g) = p_g^v(X_g) + p_g^h(X_g) \tag{12}$$

where $(X_g, X_{p_g}) \in T_{(g,p_g)}(T^*SU(2)) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)^*$. Recall the formula

$$d\beta(Y_1, Y_2) = \tilde{Y}_1(\beta(\tilde{Y}_2)) - \tilde{Y}_2(\beta(\tilde{Y}_1)) + \beta([\tilde{Y}_1, \tilde{Y}_2]) \tag{13}$$

which holds for any 1-form β on any manifold and for arbitrary choice of vector fields $\tilde{Y}_1$ and $\tilde{Y}_2$ extending the tangent vectors Y_1 and Y_2. If we use the $U_\sigma(1)$-invariant vector fields in the above formula applied to the tautological form, then the decompositions (9) and (12) and formulae (11), (4) give us

$$(\omega_c)_{(g,p_g)} = \mathcal{F}^*(\omega_{co})_{(g,p_g)} + M(q, p_q)\mathcal{F}^*(\omega_d)_{(g,p_g)} + M(g, p_g)(\omega_{\mathrm{fib}})_{(g,p_g)}. \tag{14}$$

By ω_{co} we denoted the canonical form on $T^*\mathcal{S}^2$, and ω_d is the pull-back on $T^*\mathcal{S}^2$ of the volume 2-form $\tilde{\omega}_d$ on the 2-sphere which, on the adjoint orbit $\mathcal{S}^2 \subset \mathfrak{su}(2)$, has the expression

$$(\tilde{\omega}_d)_Q(X_Q, Y_Q) = \langle Q, [X_Q Y_Q]\rangle \qquad X_Q, Y_Q \in T_Q\mathcal{S}^2 \cong Q^\perp \subset \mathfrak{su}(2).$$

The form $(\omega_{\mathrm{fib}})_{(g,p_g)}$ is the canonical form on the fibre of $\mathcal{F}$ through (g, p_g). Note that this fibre is isomorphic to $T^*U(1)$. A more detailed proof of (14) for a slightly more general situation can be found in [5].

The following proposition will constitute the first half of the proof of theorem 1.

Proposition 3. *The magnetic spherical pendulum* $(T^*S^2, \omega_c + P\omega_d, H_{sp})$ *is the symplectic quotient of the special Neumann system* $(T^*SU(2), \omega_c, H)$ *with respect to the action* $\tilde{\rho}$ *of* $U(1)$ *on* $T^*SU(2)$. *The moment map of this action is precisely the integral* $M\colon T^*SU(2) \to \mathbb{R}$ *defined in (4). The integral* $E\colon T^*SU(2) \to \mathbb{R}$ *defined in (4) induces the integral*

$$G(Q, P_Q) = -\langle [P_Q, Q], \lambda\rangle + P\langle Q, \lambda\rangle.$$

Let $(Q(t), P_Q(t))\colon [c, d] \to T^*\mathcal{S}^2$ *be a solution of the system* $(T^*\mathcal{S}^2, \omega_c + P\omega_d, H_{sp})$ *such that* $G(Q(t), P_Q(t)) = C$. *The symplectic reconstruction of* $(Q(t), P_Q(t))$ *is then every solution* $(g(t), p_g(t))\colon [c, d] \to T^*SU(2)$ *of* $(T^*SU(2), \omega_c, H)$ *such that*

$$M(g(t), p_g(t)) = P \qquad E(g(t), p_g(t)) = C \qquad t \in [c, d] \tag{15}$$

and such that $f(g(c)) = q(c)$.

If we interpret the Neumann system $(T^*SU(2), \omega_c, H)$ *as the Kaluza–Klein description of the magnetic spherical pendulum, then the integral* $M\colon T^*SU(2) \to \mathbb{R}$ *is the charge of the pendulum. The integral* $E\colon T^*SU(2) \to \mathbb{R}$ *is the sum of the angular momentum of the pendulum around its natural axis and of the gravitational potential multiplied by the charge.*

Proof. First we shall determine the moment map $\mu\colon T^*SU(2) \to \mathbb{R} \cong \mathfrak{u}(1)$ of $\tilde{\rho}$. The action $\tilde{\rho}$ is the lifting of the action ρ of $U_\sigma(1)$ on $SU(2)$. Under the right trivialization, the infinitesimal action ξ_g of $U_\sigma(1)$ at g is $\xi_g = \mathrm{Ad}_g(\sigma)$. Thus,

$$\mu(g, p_g) = p_g(\xi_g) = \left\langle p_g g^{-1}, \mathrm{Ad}_g(\sigma)\right\rangle = M(g, p_g). \tag{16}$$

From (9) we see that under the trivialization by the right translations we have

$$\mu^{-1}(P) = \left\{(g, p_g);\ p_g g^{-1} = \left\{p_g g^{-1}, \mathrm{Ad}_g(\sigma)\right\} + P(\mathrm{Ad}_g(\sigma))^\flat\right\}. \tag{17}$$

Recall that the induced symplectic form ω_{SQ} on the symplectic quotient $\mu^{-1}(P)/U_\sigma(1)$ is the 2-form satisfying the relation $i^*(\omega_c) = \pi^*(\omega_{SQ})$, where $i\colon \mu^{-1}(P) \to T^*SU(2)$ is the inclusion and $\pi\colon \mu^{-1}(P) \to \mu^{-1}(P)/U_\sigma(1)$ is the natural projection. Clearly, $\mu^{-1}(P)/U_\sigma(1) \cong T^*S^2$. If we represent the 2-sphere as the adjoint orbit $\mathcal{S}^2$, then the projection π is the restriction

$$\mathcal{F}\colon \mu^{-1}(P) \to \mu^{-1}(P)/U_\sigma(1) \cong T^*\mathcal{S}^2$$

of the map $\mathcal{F}\colon T^*SU(2) \to T^*\mathcal{S}^2$ given by (11). From (16) and (14) we now get

$$\omega_{SQ} = \omega_{co} + P\omega_d.$$

Let us now restrict the Hamiltonian $H(g, p_g) = \frac{1}{2}\|p_g g^{-1}\|^2 + \langle\lambda, \mathrm{Ad}_g(\sigma)\rangle$ to the subspace $\mu^{-1}(P)$ of $T^*SU(2)$. Since decomposition (10) is orthogonal, expression (17) gives

$$H(g, p_g) = \tfrac{1}{2}\left\|\left\{p_g g^{-1}, \mathrm{Ad}_g(\sigma)\right\}\right\|^2 + P^2 + \langle\lambda, \mathrm{Ad}_g(\sigma)\rangle.$$

Under the projection induced by the map $f(g) = \mathrm{Ad}_g(\sigma) = Q$ this function descends to the Hamiltonian of the spherical pendulum

$$H_{sp}(Q, P_Q) = \tfrac{1}{2}\|P_Q\|^2 + \langle\lambda, Q\rangle$$

where we have omitted the irrelevant constant P^2.

Clearly, the integral E of the Neumann system on $SU(2)$ is invariant with respect to the action $\tilde{\rho}$. Since the integral M is the moment map of $\tilde{\rho}$, the functions E and M Poisson-commute, as we claimed in proposition 2, and E induces an integral on the symplectic quotient. Let us denote $m_g = p_g g^{-1}$. Suppose $m_g \in \mu^{-1}(0)$. Then m_g is perpendicular to $f(g) = \mathrm{Ad}_g(\sigma)$, and $m_g \mapsto [m_g, \mathrm{Ad}_g(\sigma)]$ is the rotation through $\frac{\pi}{2}$ in $T^*_{f(g)}\mathcal{S}^2$. Thus, we can write

$$m_g = -[[m_g, \mathrm{Ad}_g(\sigma)], \mathrm{Ad}_g(\sigma)] + \langle m_g, \mathrm{Ad}_g(\sigma)\rangle \mathrm{Ad}_g(\sigma) = -[P_Q, Q] + \mu(g, p_g)Q.$$

From this we get immediately

$$G(Q, P_Q) = -\langle [P_Q, Q], \lambda\rangle + P\langle Q, \lambda\rangle. \tag{18}$$

Let the path $(g(t), p_g(t))$: $[c, d] \to T^*SU(2)$ be a symplectic reconstruction of a solution $(Q(t), P_Q(t))$: $[c, d] \to T^*\mathcal{S}^2$ of $(T^*\mathcal{S}^2, \omega_c + P\omega_d, H_{sp})$. Since $(g(t), p_g(t)) \subset \mu^{-1}(P)$, we clearly have $M(g(t), p_g(t)) \equiv P$. Any symplectic reconstruction of $(Q(t), P_Q(t))$ is of the form $\tilde\rho(u(t))(g(t), p_g(t))$ for some path $(g(t), p_g(t))$ such that $\mathcal{F}(g(t), p_g(t)) = (Q(t), P_Q(t))$. But the function E: $T^*SU(2) \to \mathbb{R}$ is invariant with respect to the action $\tilde\rho$. This, together with the decomposition

$$p_g g^{-1} = p_g^h g^{-1} + p_g^v g^{-1} = -[P_Q, Q] + PQ,$$

valid for every $p_g \in \mu^{-1}(P) = M^{-1}(P)$, finally proves the second equation of (15). □

4. Neumann system on S^2 in the axially symmetric quasimagnetic field

We shall now describe the quasimagnetic perturbation of the Neumann system on S^2 in the Lie theoretic terms. Let $J = \mathrm{diag}(1, -1)$. Then

$$\theta\colon SU(2) \to SU(2) \qquad \theta(g) = g^\theta = J \cdot g \cdot J$$

is the involution whose fixed point set is $U_\sigma(1) = \{\exp(t\sigma); t \in [0, 2\pi)\} \subset SU(2)$. The fixed point set of the map $g \mapsto (g^\theta)^{-1}$ is a copy of S^2 in $SU(2)$ which consists of the matrices $q \in SU(2)$ of the form

$$q = \begin{pmatrix} q_1 & q_2 + \mathrm{i}q_3 \\ -q_2 + \mathrm{i}q_3 & q_1 \end{pmatrix} = \begin{pmatrix} \cos\vartheta & \mathrm{e}^{\mathrm{i}\varphi}\sin\vartheta \\ -\mathrm{e}^{-\mathrm{i}\varphi}\sin\vartheta & \cos\vartheta \end{pmatrix}. \tag{19}$$

This realization of S^2 is called the Cartan model of S^2 in $SU(2)$ and we shall denote it by $\mathbb{S}^2$. For more on Cartan models of symmetric spaces, see [19, 20]. The tangent and the cotangent bundles of S^2 are of course nontrivial, but the Cartan model allows us to embed $T\mathbb{S}^2$ and $T^*\mathbb{S}^2$ in the trivial bundles $TSU(2)$ and $T^*SU(2)$, respectively. For every $q \in \mathbb{S}^2$ the map

$$\theta_q\colon \mathfrak{su}(2) \to \mathfrak{su}(2) \qquad \theta_q(\alpha) = \mathrm{Ad}_q(\mathrm{d}_e\theta(\alpha)) = \mathrm{Ad}_q(\alpha^\theta) \tag{20}$$

is an involutive isomorphism which preserves the Killing form. (By an abuse of notation we write $\theta(g) = g^\theta$ and $\mathrm{d}\theta_e(\alpha) = \alpha^\theta$ for elements $g \in SU(2)$ and $\alpha \in \mathfrak{su}(2)$ alike.) For every $q \in \mathbb{S}^2$, we have the orthogonal decomposition

$$\mathfrak{su}(2) = \mathfrak{u}_q \oplus \mathfrak{p}_q$$

into (+1)- and (−1)-eigenspaces of θ_q. Here $\mathfrak{u}_q$ is a Lie subalgebra isomorphic to $\mathfrak{u}(1)$. It is immediately clear that for every $q \in \mathbb{S}^2$ we have

$$[\mathfrak{u}_q, \mathfrak{u}_q] \subset \mathfrak{u}_q \qquad [\mathfrak{p}_q, \mathfrak{p}_q] \subset \mathfrak{u}_q \qquad [\mathfrak{u}_q, \mathfrak{p}_q] \subset \mathfrak{p}_q. \tag{21}$$

A decomposition of the Lie algebra satisfying the above relations is called the Cartan decomposition. Let $q(t)$: $[c, d] \to \mathbb{S}^2$ be a path such that $q(0) = q$. Differentiation of the identity $q(t)q(t)^\theta = e$ tells us that $q_t q^{-1} \in \mathfrak{p}_q$. Thus, the right trivialization yields the identification $T_q\mathbb{S}^2 \cong \mathfrak{p}_q \subset \mathfrak{su}(2)$. If we denote by $\mathfrak{p}_q^*$ the annihilator of $\mathfrak{u}_q$, then for every $p_q \in T_q^*\mathbb{S}^2$ we have $p_q q^{-1} \in \mathfrak{p}_q^* \subset \mathfrak{su}(2)^*$.

Let now the 1-form $\tilde\alpha_q$ on $\mathbb{S}^2$ be defined by

$$\tilde\alpha_q(X_q) = \langle \mathrm{Ad}_q(\sigma), X_q\rangle = -\langle \sigma, X_q\rangle \qquad X_q \in \mathfrak{p}_q \cong T_q\mathbb{S}^2.$$

In the second equality above we have used the facts that $\theta_q(X) = -X$, $\sigma^\theta = \sigma$ and that θ_q is an isometry. This form is obviously the pull-back by i: $\mathbb{S}^2 \to SU(2)$ of the right-invariant

1-form $\tilde{\alpha}_g$ on $SU(2)$ which takes the value σ at the identity. Using formula (13), the relation $\mathrm{d}i^*(\tilde{\alpha}_g) = i^*(\mathrm{d}\tilde{\alpha}_g)$ and the right invariance of $\tilde{\alpha}_g$ we get

$$\tilde{\omega}_h(X_q, Y_q) = \mathrm{d}\tilde{\alpha}_q(X_q, Y_q) = \langle \mathrm{Ad}_q(\sigma), [X_q, Y_q]\rangle \qquad X_q, Y_q \in \mathfrak{p}_q \cong T_q\mathbb{S}^2. \tag{22}$$

In the proof of the next proposition we shall need the expression of the natural transitive $SU(2)$-action on $SU(2)/U(1)$ in terms of the Cartan model $\mathbb{S}^2$. Observe that $\mathbb{S}^2$ is the image of the map $g \mapsto g(g^\theta)^{-1}$ of $SU(2)$ into itself. (This map is another realization of the Hopf fibration $U_\sigma(1) \hookrightarrow SU(2) \to \mathbb{S}^2$.) Therefore the natural transitive left action of $SU(2)$ on $\mathbb{S}^2$ is given by

$$\rho_g(q) = g \cdot q \cdot (g^\theta)^{-1}. \tag{23}$$

Proposition 4. *Let the motion of a particle with charge P on the sphere $S^2 \subset \mathbb{R}^3$ be governed by the homogeneous magnetic field $B_h(q) = (1, 0, 0)$. Then this motion is described by the Hamiltonian system $(T^*\mathbb{S}^2, \omega_c + P\omega_h, H_h)$, where*

$$H_h(q, p_q) = \tfrac{1}{2}\|p_q\|^2$$

and ω_h is the pull-back on $T^*\mathbb{S}^2$ of the form $\tilde{\omega}_h$, given by (22).

The Hamiltonian systems $(T^*\mathbb{S}^2, \omega_c + P\omega_h, H_h)$ and $(T^*\mathbb{S}^2, \omega_c, H_t)$, where

$$H_t(q, p_q) = \tfrac{1}{2}\|p_q\|^2 + P\left\langle p_q q^{-1}, \mathrm{Ad}_q(\sigma)\right\rangle$$

are equivalent in the sense that the Hamiltonian vector field of H_h with respect to $\omega_c + P\omega_h$ is equal to the Hamiltonian vector field of H_t with respect to ω_c.

The equation of motion of the system $(T^*\mathbb{S}^2, \omega_c + P\omega_h, H_h)$ is

$$\left(q_t q^{-1}\right)_t = P\left[q_t q^{-1}, \mathrm{Ad}_q(\sigma) + \sigma\right]. \tag{24}$$

Proof. We have to show that the 2-form ω_h gives rise to the magnetic field $B_h(q) = (1, 0, 0)$. Let $h : \mathbb{S}^2 \to \mathbb{R}$ be the function such that $(\omega_h)_q = h(q) \cdot \mathrm{d}\,vol$, where $\mathrm{d}\,vol$ is the volume 2-form on $\mathbb{S}^2$. Let $X_q, Y_q \in \mathfrak{p}_q \cong T_q\mathbb{S}^2$ be arbitrary. It is easily checked that $\mathrm{d}\,vol(X_q, Y_q) = \pm\|[X_q, Y_q]\|$. To see this, we identify $\mathbb{R}^3$ with $\mathfrak{su}(2)$ via the map $(x_1, x_2, x_3) \mapsto \mathrm{i}\sum_{j=1}^3 x_j\sigma_j$. Under this identification the bracket on $\mathfrak{su}(2)$ corresponds to the cross product on $\mathbb{R}^3$. Using expression (22), we have

$$h(q) = \left\langle \mathrm{Ad}_q(\sigma), \frac{[X_q, Y_q]}{\|[X_q, Y_q]\|}\right\rangle = \left\langle \theta_q(\mathrm{Ad}_q(\sigma)), \theta_q\left(\frac{[X_q, Y_q]}{\|[X_q, Y_q]\|}\right)\right\rangle = \left\langle \sigma, \frac{[X_q, Y_q]}{\|[X_q, Y_q]\|}\right\rangle$$

where θ_q is defined by (20). The second equality holds, because θ_q is an isometry. The third equality follows from the fact that $[X_q, Y_q] \in \mathfrak{u}_q$ whenever $X_q, Y_q \in \mathfrak{p}_q$, as stated in relations (21). Thus, besides $\theta_q(\mathrm{Ad}_q(\sigma)) = \sigma$, we also have $\theta_q([X_q, Y_q]) = [X_q, Y_q]$. Let ϑ, φ be the spherical coordinates on $\mathbb{S}^2$ and let $X_q = q_\vartheta q^{-1}$ and $Y_q = q_\varphi q^{-1}$. Expression (19) then gives

$$\frac{[X_q, Y_q]}{\|[X_q, Y_q]\|} = \begin{pmatrix} \mathrm{i}\cos(\vartheta) & -\mathrm{i}\mathrm{e}^{\mathrm{i}\varphi}\sin(\vartheta) \\ -\mathrm{i}\mathrm{e}^{-\mathrm{i}\varphi}\sin(\vartheta) & -\mathrm{i}\cos(\vartheta) \end{pmatrix}.$$

Recalling that $\sigma = \mathrm{diag}(\mathrm{i}, -\mathrm{i})$, we get $h(q(\varphi, \vartheta)) = \cos\vartheta$. Therefore,

$$(\tilde{\omega}_h)_{q(\varphi,\vartheta)} = \cos\vartheta \cdot \mathrm{d}\,vol.$$

Let B_h be a magnetic field in $\mathbb{R}^3$ and let N_q be the outward unit normal of S^2 at $q \in S^2$. The form $\tilde{\omega}_h = h \cdot \mathrm{d}\,vol$ corresponds to B_h, if $h(q) = \langle B_h(q), N_q\rangle$. The simplest field B_h such that $\langle B_h(q(\vartheta, \varphi)), N_{q(\varphi,\vartheta)}\rangle = \cos\vartheta$ is the restriction to $\mathbb{S}^2 \subset \mathbb{R}^3$ of the field $B_h(q) = (1, 0, 0)$.

Note that the form ω_h is exact. From (22) we see that $\omega_h = \mathrm{d}\alpha$, where α is the pull-back of $\tilde{\alpha}$ by $\pi\colon T^*\mathbb{S}^2 \to \mathbb{S}^2$. Consider the map

$$t_\alpha\colon T^*\mathbb{S}^2 \to T^*\mathbb{S}^2 \qquad t_\alpha(q, p_q) = (q, p_q - P\alpha_q)$$

where P is an arbitrary real constant. Let β_q be the tautological 1-form on $T^*\mathbb{S}^2$, given by $\beta_{(q,p_q)}\big(X_q^b, X_{(q,p_q)}^{ct}\big) = \big\langle p_q, X_q^b\big\rangle$ for arbitrary $\big(X_q^b, X_{(q,p_q)}^{ct}\big) \in T_{(q,p_q)}(T^*\mathbb{S}^2)$. Then we obviously have $t_\alpha^*(-\beta) = -\beta + P\alpha$. Thus,

$$t_\alpha^*(\omega_c) = \mathrm{d}t_\alpha^*(-\beta) = \mathrm{d}(-\beta + P\alpha) = \omega_c + P\omega_h.$$

On the other hand,

$$t_\alpha^*(H_t) = H_h$$

which proves the second part of the proposition. For more information on the argument used above, we refer the reader to [4, 21].

It remains to prove that equation (24) is the equation of motion of our system. By means of the Legendre transformation we see that the solutions of $(T^*\mathbb{S}^2, \omega_c, H_t)$ are the extremals $q(t)\colon [c, d] \to \mathbb{S}^2$ of the Lagrangian

$$\mathcal{L}_h(q(t)) = \int_c^d \left(\frac{1}{2}\left\|q_t q^{-1}\right\| - P\left\langle q_t q^{-1}, \mathrm{Ad}_q(\sigma)\right\rangle\right) \mathrm{d}t.$$

Let $a(t, s)\colon [c, d] \times (-\epsilon, \epsilon) \to SU(2)$ be a smooth map such that $a(t, 0) \equiv e$, $a(c, s) \equiv q(a)$ and $a(d, s) \equiv q(b)$. Then $q(t, s) = \rho_{a(t,s)}(q(t)) = a(t, s)q(t)(a(t, s)^\theta)^{-1}$ is a family of paths in $\mathbb{S}^2$ connecting $q(a)$ and $q(b)$. From the transitivity of action (23) it follows that every such family is of this form for some $a(t, s)$. In a similar way as in proposition 2 we get

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathcal{L}_h(q(t, s)) = \int_c^d \big(\big\langle\mathcal{E} - \mathrm{Ad}_{q(t)}(\mathcal{E}^\theta), \delta a\big\rangle\big)\, \mathrm{d}t$$

where

$$\mathcal{E} = \left(q_t q^{-1}\right)_t - P\left[q_t q^{-1}, \mathrm{Ad}_q(\sigma)\right]$$

and $\delta a = \frac{\mathrm{d}}{\mathrm{d}s} a(t, s)a(t, s)^{-1}\big|_{s=0}$ is the variation. Therefore, for $q(t)$ to be an extremal, $\pi_q(\mathcal{E}) = \mathcal{E} - \mathrm{Ad}_q(\mathcal{E}^\theta)$ has to be equal to zero. For every $q \in \mathbb{S}^2$, the expression $\pi_q(\mathcal{E})$ is the orthogonal projection of $\mathcal{E}$ on $\mathfrak{p}_q \cong T_q\mathbb{S}^2 \subset \mathfrak{su}(2)$. The fact that $\mathbb{S}^2$ is a totally geodesic submanifold of $SU(2)$ and a simple calculation show that $\pi_q(\mathcal{E}) = (q_t q^{-1})_t - [q_t q^{-1}, \mathrm{Ad}_q(\sigma) + \sigma]$, which proves that (24) is indeed the equation of motion of our system. □

We now come to the description, first in the Lagrangian terms, of the Neumann system perturbed by the quasimagnetic field B described in the introduction.

Proposition 5. *Let the quadratic potential $V(q)\colon S^2 \to \mathbb{R}$ have arbitrary eigenvalues $\{\alpha, \beta, \gamma\}$. Let the motion of a charged particle on the sphere $S^2 \subset \mathbb{R}^3$ be governed by the potential force $F(q) = \mathrm{grad}(V(q))$ and by the Lorentz-type force $L(q, q_t) = \langle q_t \times q, (1, 0, 0)\rangle \cdot (q_t \times (1, 0, 0))$. Then, in terms of the Cartan model, the Lagrangian of this system is*

$$\mathcal{L}_m(q(t)) = \int_a^b \left(\frac{1}{2}\big(\left\|q_t q^{-1}\right\|^2 - \big\langle q_t q^{-1}, \mathrm{Ad}_q(\sigma)\big\rangle^2\big) - \langle\lambda, \mathrm{Ad}_q(\sigma)\rangle\right) \mathrm{d}t \quad (25)$$

for a suitably chosen $\lambda \in \mathfrak{su}(2)$.

Proof. Let us find the Euler–Lagrange equation of the Lagrangian $\mathcal{L}_m$. As in the proof of proposition 4, let $a(t, s)\colon [c, d] \times (-\epsilon, \epsilon) \to SU(2)$ be a smooth map such that $a(t, 0) \equiv e$,

$a(c,s) \equiv q(a)$ and $a(d,s) \equiv q(b)$. Let again $q(t,s) = a(t,s)q(t)(a(t,s)^\theta)^{-1}$ be a family of paths joining $q(a)$ and $q(b)$. This time the variation gives

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathcal{L}_m(q(t,s)) = \int_c^d \left(\langle \mathcal{G} - \mathrm{Ad}_{q(t)}(\mathcal{G}^\theta), \delta a\rangle\right) \mathrm{d}t$$

where

$$\mathcal{G} = \left(q_t q^{-1}\right)_t - \left\langle q_t q^{-1}, \mathrm{Ad}_q(\sigma)\right\rangle \cdot \left[q_t q^{-1}, \mathrm{Ad}_q(\sigma)\right] - [\lambda, \mathrm{Ad}_q(\sigma)].$$

Similarly as before, we see that

$$\pi_q\left(\left\langle q_t q^{-1}, \mathrm{Ad}_q(\sigma)\right\rangle \cdot \left[q_t q^{-1}, \mathrm{Ad}_q(\sigma)\right]\right) = \left\langle q_t q^{-1}, \mathrm{Ad}_q(\sigma)\right\rangle \cdot \left[q_t q^{-1}, \mathrm{Ad}_q(\sigma) + \sigma\right].$$

Let

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Representation (19) of S^2 as the Cartan model $\mathbb{S}^2 \subset SU(2)$ sends the vector $K(q_t \times q)$ into the matrix $q_t q^{-1}$. Recall that $\langle q_t q^{-1}, \mathrm{Ad}_q(\sigma)\rangle = -\langle q_t q^{-1}, \sigma\rangle$, since θ_q is an isometry. It follows now from the third part of proposition 4 that the Lorentz-type force of the system given by $\mathcal{L}_m$ is indeed equal to $\langle \dot{q} \times q, (1,0,0)\rangle(\dot{q} \times (1,0,0))$.

Finally we have to show that for a suitably chosen $\lambda \in \mathfrak{su}(2)$ the gradient field of the function $\langle \lambda, \mathrm{Ad}_q(\sigma)\rangle = -\frac{1}{2}\mathrm{Tr}(\lambda \cdot \mathrm{Ad}_q(\sigma))$ is equal to the gradient field grad$(V(q))$, where $V(q)\colon \mathbb{S}^2 \to \mathbb{R}$ is a quadratic form with arbitrary eigenvalues $\{\alpha, \beta, \gamma\}$. If $\delta = -\frac{1}{2}(\beta + \gamma)$, then the quadratic form $V(q) + \delta\left(q_1^2 + q_2^2 + q_3^2\right)$ has eigenvalues $\{a, b, -b\}$, where $a = \alpha + \delta$ and $b = \beta + \delta$. The function $\delta\left(q_1^2 + q_2^2 + q_3^2\right)$ is constant on S^2, therefore its gradient is equal to zero. We can assume, without loss of generality, that the eigenvalues of the potential $V(q)$ are a set of the form $\{a, b, -b\}$. Since $q \in \mathbb{S}^2 \subset SU(2)$, we have $q^{-1} = q^\theta = \mathrm{Ad}_J(q)$. Matrix multiplication and evaluation of the trace then give

$$\langle \lambda, \mathrm{Ad}_q(\sigma)\rangle = \lambda_1\left(q_1^2 - q_2^2 - q_3^2\right) - 2\lambda_3 q_1 q_2 + 2\lambda_2 q_1 q_3$$

where again $\lambda = \mathrm{i}(\lambda_1\sigma_1 + \lambda_2\sigma_2 + \lambda_3\sigma_3)$ and σ_i are Pauli matrices. Let A be the symmetric 3×3 matrix of the quadratic form $\langle \lambda, \mathrm{Ad}_q(\sigma)\rangle$. The characteristic equation of A is

$$\det(A - zI) = -(\lambda_1 + z) \cdot \left(z^2 - \left(\lambda_1^2 + 4\lambda_2^2 + 4\lambda_3^2\right)\right) = 0.$$

From this we see that $\lambda \in \mathfrak{su}(2)$ will yield the quadratic form with the desired eigenvalues, if $\lambda_1 = a$ and $\lambda_1^2 + 4\lambda_2^2 + 4\lambda_3^2 = b^2$. □

We shall now describe the relation between the system given by $\mathcal{L}_m$ and the special Neumann system on $S^3 = SU(2)$. We shall use a projection method similar to the one described by Olshanetsky and Perelomov in [22]. First observe that every element $g \in SU(2)$ can be written in the form $g = q \cdot u$, where $q \in \mathbb{S}^2$ and $u \in U_\sigma(1)$. For elements g which are not of the form

$$g = \begin{pmatrix} 0 & a \\ -\bar{a} & 0 \end{pmatrix} \tag{26}$$

there are precisely two such decompositions, namely $g = q \cdot u = (-q) \cdot (-u)$. Elements of the form (26) comprise an equatorial 1-sphere in $\mathbb{S}^2$. For every such element g and every $u \in U_\sigma(1)$ there exists $q_u \in \mathbb{S}^2$ such that $g = q_u \cdot u$. In other words, for a general $g \in SU(2)$ the fibre $\{gu;\, u \in U_\sigma(1)\} \cong S^1$ of the Hopf map $g \mapsto \mathrm{Ad}_g(\sigma)$ intersects the Cartan model $\mathbb{S}^2$ in two antipodal points, while for g of the form (26) the whole fibre lies in $\mathbb{S}^2$. For a proof of an analogous claim for a general Cartan model, see [19].

Let $g\colon [c,d] \to SU(2)$ be a path and let

$$g(t) = q(t)\cdot u(t) \qquad q(t)\colon [c,d] \to \mathbb{S}^2 \qquad u(t)\colon [c,d] \to U_\sigma(1) \tag{27}$$

be its decomposition in the sense described above. Since $u_t u^{-1} = r(t)\sigma$ and $\mathrm{Ad}_g(\sigma) = \mathrm{Ad}_q(\sigma)$, we have

$$g_t g^{-1} = q_t q^{-1} + \mathrm{Ad}_q\left(u_t u^{-1}\right) = q_t q^{-1} + r_t\,\mathrm{Ad}_q(\sigma) \tag{28}$$

where $r(t)\colon [c,d] \to \mathbb{R}$ is a real function.

Proposition 6. *Let $g(t)\colon [a,d] \to SU(2)$ be a solution of the special Neumann system $(T^*SU(2), \omega_c, H)$. Let $g(t) = q(t)u(t)$ be its decomposition of the form (27). Then the path $q(t)\colon [c,d] \to \mathbb{S}^2$ is an extremal of the Lagrangian $\mathcal{L}_m$ given by (25).*

Proof. In the proof of proposition 2 we have seen that the Lagrangian of the special Neumann system is

$$\mathcal{L}(g(t)) = \int_c^d \left(\frac{1}{2}\left\|g_t g^{-1}\right\|^2 - \langle\lambda, \mathrm{Ad}_g(\sigma)\rangle\right)\mathrm{d}t.$$

In terms of decomposition (28) this gives

$$\mathcal{L}(q(t), r(t)) = \int_c^d \left(\frac{1}{2}\left\|q_t q^{-1} + r_t\,\mathrm{Ad}_q(\sigma)\right\|^2 - \langle\lambda, \mathrm{Ad}_q(\sigma)\rangle\right)\mathrm{d}t. \tag{29}$$

Recall that the function $M\colon T^*SU(2) \to \mathbb{R}$ given by (4) is an integral of the special Neumann system. This means that, along a solution $g(t)$ of this system, we have

$$\left\langle g_t g^{-1}, \mathrm{Ad}_g(\sigma)\right\rangle = \left\langle q_t q^{-1} + r_t \mathrm{Ad}_q(\sigma), \mathrm{Ad}_q(\sigma)\right\rangle = \left\langle q_t q^{-1}, \mathrm{Ad}_q(\sigma)\right\rangle + r_t = P \tag{30}$$

where P is a constant. If we put $r_t = P - \langle q_t q^{-1}, \mathrm{Ad}_q(\sigma)\rangle$ into expression (29), a short calculation shows that the path $q(t)\colon \mathbb{S}^2 \to \mathbb{R}$ is an extremal of the Lagrangian $\mathcal{L}_m + P^2$, whenever the path $g(t) = q(t)u(t)$ is an extremal for $\mathcal{L}$, for a suitable $u(t)$. □

We shall now describe the system given by $\mathcal{L}_m$ in the Hamiltonian terms and prove its integrability. For this purpose we shall derive a new expression for the equation of motion by means of the projection method. Let, as before, $p_q = (q_t)^\flat$, where $\flat\colon T_q\mathbb{S}^2 \to T_q^*\mathbb{S}^2$ is the map given by $(q_t)^\flat = \langle q_t, -\rangle$ and $\langle -, -\rangle$ is the natural metric on $\mathbb{S}^2$. Formulae (28) and (30) show immediately that the extremals of the Lagrangian $\mathcal{L}_m + P^2$ are the solutions of the Hamiltonian system $(T^*\mathbb{S}^2, \omega_c, H_m)$, where the Hamiltonian H_m is given by

$$H_m(q, p_q) = \tfrac{1}{2}\left\|p_q q^{-1} + \left(P - \left\langle p_q q^{-1}, \mathrm{Ad}_q(\sigma)\right\rangle\right)(\mathrm{Ad}_q(\sigma))^\flat\right\|^2 + \langle\lambda, \mathrm{Ad}_q(\sigma)\rangle.$$

We have already noted in the proof of proposition 5 that $q_t q^{-1}$ corresponds to $K(q_t \times q)$. Using this and the relation $\langle q_t q^{-1}, \mathrm{Ad}_q(\sigma)\rangle = -\langle q_t q^{-1}, \sigma\rangle$, we see that H_m can indeed be written in the form (1).

Theorem 2. *The equation of motion of the system $(T^*\mathbb{S}^2, \omega_c, H_m)$ can be given in the form*

$$\begin{aligned}\left(q_t q^{-1}\right)_t &= \left\langle\left(q_t q^{-1}\right)_t, \mathrm{Ad}_q(\sigma)\right\rangle \mathrm{Ad}_q(\sigma)\\ &\quad + \left(\left\langle q_t q^{-1}, \mathrm{Ad}_q(\sigma)\right\rangle - P\right)\left[q_t q^{-1}, \mathrm{Ad}_q(\sigma)\right] + [\lambda, \mathrm{Ad}_q(\sigma)].\end{aligned} \tag{31}$$

This equation is equivalent to the Lax equation

$$L_t = [A, L]$$

where the Lax pair $(L(z), A(z))$ is given by

$$\begin{aligned}L(z) &= \mathrm{Ad}_q(\sigma) + z\left(q_t q^{-1} + \left(P - \left\langle q_t q^{-1}, \mathrm{Ad}_q(\sigma)\right\rangle\right)\mathrm{Ad}_q(\sigma)\right) + z^2\lambda\\ A(z) &= \left(q_t q^{-1} + \left(P - \left\langle q_t q^{-1}, \mathrm{Ad}_q(\sigma)\right\rangle\right)\mathrm{Ad}_q(\sigma)\right) + z\lambda.\end{aligned} \tag{32}$$

The function $F\colon T^*\mathbb{S}^2 \to \mathbb{R}$ *given by*

$$F(q, p_q) = \left\langle p_q q^{-1}, \lambda\right\rangle + \left(\left\langle p_q q^{-1}, \sigma\right\rangle + P\right) \langle \lambda, \mathrm{Ad}_q(\sigma)\rangle \tag{33}$$

is an integral of $(T^*\mathbb{S}^2, \omega_c, H_m)$*. If we put* $l = (\lambda_1, -\lambda_3, \lambda_2)$*, then expression (33) is equal to integral (2).*

Proof. The solutions of the system $(T^*\mathbb{S}^2, \omega_c, H_m)$ are the extremals $q(t)\colon T^*\mathbb{S}^2 \to \mathbb{R}$ of the Lagrangian $\mathcal{L}_m + P^2$. These, in turn, are the projections of the extremals of $\mathcal{L}\colon TSU(2) \to \mathbb{R}$ to $\mathbb{S}^2 \subset SU(2)$, as we have seen in proposition 6. The Euler–Lagrange equation of $\mathcal{L}$ is $(g_t g^{-1})_t = [\lambda, \mathrm{Ad}_g(\sigma)]$. Suppose the integral $M(g, p_g)\colon T^*SU(2) \to \mathbb{R}$ takes the value P along our solution. Then, in the decomposition $g_t g^{-1} = q_t q^{-1} + r_t \mathrm{Ad}_q(\sigma)$, we have $r_t = P - \langle q_t q^{-1}, \mathrm{Ad}_q(\sigma)\rangle$. Putting this into the above Euler–Lagrange equation yields equation (31).

A straightforward check shows that the Lax equation for Lax pair (32) is equivalent to equation (31). The same argument as in the proof of proposition 2 shows that the coefficients of the polynomial $\langle L(z), L(z)\rangle$ are integrals of the motion of $(T^*\mathbb{S}^2, \omega_c + P\omega_h, H_n)$. The z^3-coefficient is the function given by (33).

In the proof of proposition 5 we have seen that representation (19) sends the vector $K(p_q \times q)$ into the element $p_q q^{-1} \in \mathfrak{su}(2)^*$. Since $\lambda = \sigma_1\lambda_1 + \sigma_2\lambda_2 + \sigma_3\lambda_3$, it is now clear that expressions (33) and (2) indeed represent the same function. □

Proposition 7. *Let* $g(t)\colon [c, d] \to SU(2)$ *be a solution of the system* $(T^*SU(2), \omega_c, H)$ *such that*

$$M(g(t), p_g(t)) = P \qquad E(g(t), p_g(t)) = C \qquad t \in [c, d].$$

Let $g(t) = q(t)u(t)$ *be its decomposition of the form (27). Then* $q(t)\colon [c, d] \to \mathbb{S}^2$ *is a solution of the system* $(T^*\mathbb{S}^2, \omega_c, H_m)$ *such that*

$$F\left(q(t), (q_t(t))^\flat\right) = C \qquad t \in [c, d].$$

Proof. Let $g(t)\colon [c, d] \to SU(2)$ be our solution. Then $g(t)$ is a solution of equation (5). We have seen in the proposition above that the $\mathbb{S}^2$-part of the decomposition $g(t) = q(t)u(t)$ solves equation (31) and is therefore a solution of the system $(T^*\mathbb{S}^2, \omega_c, H_m)$.

By definition we have $E(g(t), (g_t(t))^\flat) = \langle g_t g^{-1}, \lambda\rangle$. Using the expression $g_t g^{-1} = q_t q^{-1} + r_t \mathrm{Ad}_q(\sigma)$ and the relation $r_t = P - \langle q_t q^{-1}, \mathrm{Ad}_q(\sigma)\rangle$ again, we finally get

$$E(g(t), p_g(t)) = \left\langle q_t q^{-1}, \lambda\right\rangle + \left(P - \left\langle q_t q^{-1}, \mathrm{Ad}_q(\sigma)\right\rangle\right)\langle \lambda, \mathrm{Ad}_q(\sigma)\rangle.$$

Since $\langle q_t q^{-1}, \mathrm{Ad}_q(\sigma)\rangle = -\langle q_t q^{-1}, \sigma\rangle$, we have $E(g(t), p_g(t)) = F(q(t), (q_t(t))^\flat)$, which concludes the proof of the proposition. □

Proof of theorem 1. Let $(Q(t), P_Q(t))\colon [c, d] \to T^*\mathcal{S}^2$ be a solution of the magnetic spherical pendulum $(T^*\mathcal{S}^2, \omega_c + P\omega_d, H_{sp})$ such that

$$G(Q(t), P_Q(t)) = C \qquad t \in [c, d].$$

In proposition 3 we have seen that the symplectic reconstruction $(g(t), p_g(t))\colon [c, d] \to SU(2)$ with a chosen initial point in the fibre $\mathcal{F}^{-1}(Q(c), P_Q(c))$ is a solution of the system $(T^*SU(2), \omega_c, H)$ such that

$$M(g(t), p_g(t)) = P \qquad E(g(t), p_g(t)) = C \qquad t \in [c, d].$$

In the spherical coordinates the decomposition $g(t) = q(t)u(t)$ has the form

$$g(t) = \begin{pmatrix} e^{i\psi(t)}\cos\vartheta(t) & e^{i\varphi(t)}\sin\vartheta(t) \\ -e^{-i\varphi(t)}\sin\vartheta(t) & e^{-i\psi(t)}\cos\vartheta(t) \end{pmatrix}$$
$$= \begin{pmatrix} \cos\vartheta(t) & e^{i(\varphi(t)+\psi(t))}\sin\vartheta(t) \\ -e^{-i(\varphi(t)+\psi(t))}\sin\vartheta(t) & \cos\vartheta(t) \end{pmatrix} \begin{pmatrix} e^{i\psi(t)} & 0 \\ 0 & e^{-i\psi(t)} \end{pmatrix} \tag{34}$$

We have shown in proposition 7 that the first factor above is a solution $q(t) : [c, d] \to \mathbb{S}^2$ of $(T^*\mathbb{S}^2, \omega_c + P\omega_h, H_n)$ such that

$$F(q(t), (\dot{q}(t))^\flat) = C \qquad t \in [c, d].$$

If we compare formula (8) which relates the expressions of $g(t)$ and $Q(t)$ in spherical coordinates to formula (34), we see that the path $(q(t), (q_t(t))^\flat)$: $[c, d] \to T^*\mathbb{S}^2$, where

$$q(t) = \left(\cos\left(\tfrac{1}{2}\vartheta\right), e^{i(\varphi(t)-\frac{\pi}{2})}\sin\left(\tfrac{1}{2}\vartheta(t)\right)\right)$$

is indeed a solution of $(T^*\mathbb{S}^2, \omega_c, H_m)$, if

$$Q(t) = (\cos\vartheta(t), e^{i\varphi(t)}\sin(\vartheta(t))). \qquad \square$$

Remark 1. The Hamiltonian (1) can be expressed in the form

$$H(q, p_q) = \tfrac{1}{2}\left(\|p_q\|^2 - \langle p_q \times \sigma\rangle^2 + P^2\right) + V(q).$$

Let U_0 be the total energy of a solution $(q(t), p_q(t))$ of $(T^*\mathbb{S}^2, \omega_c, H_m)$ obtained from a solution of the magnetic spherical pendulum with zero charge and let U_P be the energy of the solution $(\tilde{q}(t), \tilde{p}_q(t))$ of $(T^*\mathbb{S}^2, \omega_c, H_m)$ obtained from a solution of the magnetic spherical pendulum with charge P. Suppose $(q(t), p_q(t))$ and $(\tilde{q}(t), \tilde{p}_q(t))$ have the same initial conditions. The above expression of H_m shows that $U_P = U_0 + P^2$.

We conclude the paper by a brief description of the special case, where the potential $V: S^2 \to \mathbb{R}$ is axially symmetric, say of the form $V(q) = aq_1^2 + bq_2^2 + bq_3^2$. It follows from the proof of proposition 5 that in suitable coordinates and in terms of the Cartan model, such a potential can be written in the form $V(q) = \langle\sigma, \mathrm{Ad}_q(\sigma)\rangle$ up to an irrelevant additive constant.

Let $(T^*SU(2), \omega_c, H_s)$ be the special Neumann system on $SU(2)$ with the Hamiltonian $H_s(g, p_g) = \frac{1}{2}\|p_g\|^2 + \langle\sigma, \mathrm{Ad}_g(\sigma)\rangle$. The integrals of this system are

$$M_s(g, p_g) = \left\langle p_g g^{-1}, \mathrm{Ad}_g(\sigma)\right\rangle \qquad E_s(g, p_g) = \left\langle p_g g^{-1}, \sigma\right\rangle. \tag{35}$$

Let, as before, $U_\sigma(1) = \{\exp(t\sigma), t \in [0, 2\pi)\} \subset SU(2)$ act on $(T^*SU(2), \omega_c, H_s)$. Recall that the symplectic reduction at the level P is the magnetic spherical pendulum $(T^*S^2, \omega_c + P\omega_d, H_{sp})$, where

$$H_{sp}(Q, P_Q) = \tfrac{1}{2}\|P_Q\|^2 + \langle Q, S\rangle \qquad S = (1, 0, 0).$$

An additional integral of this system is

$$G_s(Q, P_Q) = -\langle P_q \times Q, S\rangle + P\langle Q, S\rangle.$$

The additional symmetry does not change much here.

On the other hand, the projection procedure from the system on $SU(2)$ to the system on the Cartan model $\mathbb{S}^2 \subset SU(2)$ simplifies considerably. The Hamiltonian H_{sm}: $T^*\mathbb{S}^2 \to \mathbb{R}$ of the quasimagnetic system obtained by projection on $T^*\mathbb{S}^2$ has the form

$$H_{sm}(q, p_q) = \tfrac{1}{2}\left\| p_q q^{-1} + \left(P - \left\langle p_q q^{-1}, \mathrm{Ad}_q(\sigma)\right\rangle\right)(\mathrm{Ad}_q(\sigma))^\flat\right\|^2 + \langle\sigma, \mathrm{Ad}_q(\sigma)\rangle.$$

The action ρ of the group $U_\sigma(1)$ on the Cartan model $\mathbb{S}^2$ is given by $\rho_u(q) = u \cdot q \cdot (u^\theta)^{-1} = \mathrm{Ad}_u(q)$. The Hamiltonian H_{sm} is invariant with respect to this action. The corresponding moment map μ: $T^*\mathbb{S}^2 \to \mathbb{R}$ is

$$\mu(q, p_q) = \langle p_q q^{-1}, \mathrm{Ad}_q(\sigma)\rangle.$$

Proposition 8. *Let the Hamiltonian function* H_{hm}: $T^*\mathbb{S}^2 \to \mathbb{R}$ *be given by*

$$H_{hm}(q, p_q) = \tfrac{1}{2}\|p_q\|^2 + R\langle p_q q^{-1}, \mathrm{Ad}_q(\sigma)\rangle + \langle \sigma, \mathrm{Ad}_q(\sigma)\rangle.$$

A path $\gamma(t)$: $[c, d] \to T^*\mathbb{S}^2$ *such that* $\mu(\gamma(t)) \equiv D$ *is a solution of the system* $(T^*\mathbb{S}^2, \omega_c, H_{sm})$ *if and only if it is a solution of the system* $(T^*\mathbb{S}^2, \omega_c, H_{hm})$ *for* $R = P - D$.

Proof. Clearly, the Hamiltonian H_{hm}: $T^*\mathbb{S}^2 \to \mathbb{R}$ is also invariant with respect to the action ρ. Let $D \in \mu(T^*\mathbb{S}^2)$. After restricting to the level set $\mu^{-1}(D) \subset T^*\mathbb{S}^2$, the Hamiltonian H_{sm}: $T^*\mathbb{S}^2 \to \mathbb{R}$ becomes

$$H_{sm}(q, p_q) = \tfrac{1}{2}\left\| p_q q^{-1} + (P - D)\mathrm{Ad}_q(\sigma)^\flat \right\|^2 + \langle \sigma, \mathrm{Ad}_q(\sigma)\rangle.$$

If we set $R = P - D$, a short calculation shows that we have

$$H_{sm}(q, p_q) = H_{hm}(q, p_q) - R^2 \qquad \text{for every } (q, p_q) \in \mu^{-1}(D) \subset T^*\mathbb{S}^2.$$

Thus, the symplectic quotient $\left(\mu^{-1}(D)/U_\sigma(1), \omega_{sq}, H^q_{sm}\right)$ of $(T^*\mathbb{S}^2, \omega_c, H_{sm})$ is equal to the symplectic quotient $\left(\mu^{-1}(D)/U_\sigma(1), \omega_{sq}, H^q_{hm}\right)$ of $(T^*\mathbb{S}^2, \omega_c, H_{hm})$.

Let, as before,

$$q = \begin{pmatrix} \cos\vartheta & \mathrm{e}^{\mathrm{i}\varphi}\sin\vartheta \\ -\mathrm{e}^{-\mathrm{i}\varphi}\sin\vartheta & \cos\vartheta \end{pmatrix}$$

be the parametrization of $\mathbb{S}^2$ with the spherical coordinates. In these coordinates the action ρ of $U_\sigma(1)$ is given by $\rho_s(\varphi, \vartheta) = (\varphi + s, \vartheta)$. Observe that the symplectic quotient space $\left(\mu^{-1}(D)/U_\sigma(1), \omega_{sq}\right)$ is equal to the cotangent bundle (T^*K, ω_c) of the interval $K = \{\vartheta; \vartheta \in [0, \pi]\}$ and ω_c is the canonical cotangent form.

Let the path

$$\beta(t) = (\vartheta(t), p_\vartheta(t)): [c, d] \to T^*K$$

be a solution of the system $\left(\mu^{-1}(D)/U_\sigma(1), \omega_{sq}, H^q_{sm}\right) = \left(\mu^{-1}(D)/U_\sigma(1), \omega_{sq}, H^q_{hm}\right)$ with the initial condition $(\vartheta(c), p_\vartheta(c)) = (a_1, b_1)$. We will construct the symplectic reconstruction of $\beta(t)$ with respect to $(T^*\mathbb{S}^2, \omega_c, H_{sm})$ with the initial condition $(a_1, a_2, b_1, b_2) \in \mu^{-1}(D) \subset T^*\mathbb{S}^2$. In terms of the spherical coordinates the moment map μ: $T^*\mathbb{S}^2 \to \mathbb{R}$ has the expression

$$\mu(\varphi, \vartheta, p_\varphi, p_\vartheta) = -2p_\varphi \sin^2\vartheta. \tag{36}$$

Since $p_\varphi = \dot\varphi$, the symplectic reconstruction $\gamma(t)$ is given by

$$\gamma(t) = (\varphi(t), \vartheta(t), p_\varphi(t), p_\vartheta(t)) = \left(-\frac{D}{2}\int \frac{1}{\sin^2\vartheta(t)}\,\mathrm{d}t, \vartheta(t), -\frac{D}{2\sin^2\vartheta(t)}, p_\vartheta(t)\right) \tag{37}$$

where $(\vartheta(t), p_\vartheta(t)) = \beta(t)$. Obviously, the reconstruction of $\beta(t)$ with respect to the system $(T^*\mathbb{S}^2, \omega_c, H_{hm})$ is given by the same formula. Let $(a_1, a_2, b_1, b_2) \in \mu^{-1}(D) \subset T^*\mathbb{S}^2$ be arbitrary. Let $\gamma(t)$ be the solution of $(T^*\mathbb{S}^2, \omega_c, H_{sm})$ such that $\gamma(c) = (a_1, a_2, b_1, b_2)$ and let $\tilde\gamma(t)$ be a solution of $(T^*\mathbb{S}^2, \omega_c, H_{hm})$ such that $\tilde\gamma(c) = \gamma(c)$. Then we have $\tilde\gamma(t) \equiv \gamma(t)$. Since the choice of the initial condition $(a_1, a_2, b_1, b_2) \in \mu^{-1}(D)$ was arbitrary, the proposition is proved. □

Remark 2. In spite of its appearence, solution (37) is not singular. Let $\gamma(t) = (\varphi(t), \vartheta(t), p_\varphi(t), p_\vartheta(t))$ be a solution such that $\mu(\gamma(t)) \equiv D \neq 0$. Suppose $\vartheta(t)$ could approach 0 or π, then expression (36) shows that $\|p_\varphi(t)\|$ should grow to infinity. This would also push the Hamiltonian H_{hm} to infinity. Since H_{hm} is constant along the solution $\gamma(t)$, the value $\vartheta(t)$ can approach 0 or π only if $D = 0$.

It follows from proposition 4 that the system $(T^*\mathbb{S}^2, \omega_c, H_{hm})$ is equivalent to the system $\left(T^*\mathbb{S}^2, \omega_c + R\omega_h, H_c\right)$, where the form ω_h is given by (22) and

$$H_c(q, p_q) = \tfrac{1}{2}\|p_q\|^2 + \langle\sigma, \mathrm{Ad}_q(\sigma)\rangle.$$

It also follows from proposition 4 that the system $(T^*\mathbb{S}^2, \omega_c, H_c)$ describes the motion of a particle with charge R under the influence of the potential $V(q) = aq_1^2 + bq_2^2 + bq_3^2$ and the homogeneous magnetic field $B_h(q) = (1, 0, 0)$. The above discussion proves the following corollary of theorem 1.

Corollary 1. *Let the curve* $(Q(t), P_Q(t))\colon [c, d] \to T^*S^3$ *be a solution of the magnetic spherical pendulum* $\left(T^*S^2, \omega_c + P\omega_d, H_{sp}\right)$ *such that* $G(Q(t), P_Q(t)) = C$ *for every* $t \in [c, d]$. *If*

$$Q(t) = (\cos(\vartheta(t)), \mathrm{e}^{\mathrm{i}\varphi(t)}\sin(\vartheta(t))$$

then the curve

$$q(t) = \left(\cos\left(\tfrac{1}{2}\vartheta(t)\right), \mathrm{e}^{\mathrm{i}(\varphi(t) - \frac{\pi}{2})}\sin\left(\tfrac{1}{2}\vartheta(t)\right)\right)$$

is a solution of the axisymmetric Neumann system $\left(T^*\mathbb{S}^2, \omega_c + P\omega_h, H_c\right)$ *describing a particle with charge* P *moving under the influence of the potential* $V(q)$ *and the homogeneous magnetic field* $B_h(q) = (1, 0, 0)$. *We have* $F(q(t), (q_t(t))^\flat) = C$ *along this solution.*

5. Summary

In this paper we established a relation between the charged spherical pendulum in the magnetic field of the Dirac monopole and a quasimagnetic perturbation of the Neumann system. Claims analogous to theorems 1 and 2 should hold, if we replace S^2 by an arbitrary Hermitian symmetric space, compact or non-compact. The essential part of our construction is the fact that the sphere S^2 can be represented as the adjoint orbit in $\mathfrak{su}(2)$ *and* as the Cartan model in $SU(2)$. The Riemannian manifolds which can be represented in these two ways are precisely the Hermitian symmetric spaces. Of course, in that general setting one loses the advantage of the easy use of coordinates.

The relation between the magnetic spherical pendulum and the perturbed Neumann system is particularly simple in the case where the quadratic Neumann potential is axially symmetric. In this case we obtained the relation between the magnetic spherical pendulum and the Neumann system perturbed by the topologically trivial homogeneous magnetic field $B_h(1, 0, 0)$. In [23] the authors give a thorough description of the geometric quantization of the axially symmetric Neumann system. It seems that it would be quite easy to extend their results to the axially symmetric Neumann system with the homogeneous magnetic field. Our construction could therefore be used to shed some new light on the geometric quantization of the charged spherical pendulum in the field of the magnetic monopole.

Acknowledgments

The author wishes to express his gratitude to the first referee whose valuable remarks and suggestions helped to improve and sharpen some statements and to avoid some erroneous ones. The research for this paper was supported in part by the Ministry of Science and Technology of the Republic of Slovenia.

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